

# ALMOST INVARIANT HALF-SPACES OF ALGEBRAS OF OPERATORS

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**ABSTRACT.** Given a Banach space  $X$  and a bounded linear operator  $T$  on  $X$ , a subspace  $Y$  of  $X$  is almost invariant under  $T$  if  $TY \subseteq Y + F$  for some finite-dimensional “error”  $F$ . In this paper, we study subspaces that are almost invariant under every operator in an algebra  $\mathfrak{A}$  of operators acting on  $X$ . We show that if  $\mathfrak{A}$  is norm closed then the dimensions of “errors” corresponding to operators in  $\mathfrak{A}$  must be uniformly bounded. Also, if  $\mathfrak{A}$  is generated by a finite number of commuting operators and has an almost invariant half-space (that is, a subspace with both infinite dimension and infinite codimension) then  $\mathfrak{A}$  has an invariant half-space.

## 1. INTRODUCTION

The notion of an almost invariant subspace was recently introduced in [APTT]. If  $T$  is an operator on a Banach space  $X$  then a subspace  $Y$  of  $X$  is called ***almost invariant under  $T$***  if there exists a finite-dimensional subspace  $F$  of  $X$  such that

$$(1) \quad TY \subseteq Y + F.$$

Clearly, if  $Y$  has finite dimension or finite codimension then  $Y$  is almost invariant under every operator on  $X$ .

**Definition 1.1.** [APTT] A subspace  $Y \subseteq X$  is called a ***half-space*** if  $Y$  is both of infinite dimension and of infinite codimension.

The question whether every operator on a Banach space has an almost invariant half-space was posed in [APTT]; it was solved there for certain classes of operators. Just as the studies of transitive algebras

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generalize the Invariant Subspace Problem for a single operator, the purpose of this paper is to introduce and study the notion of a subspace that is simultaneously *almost* invariant under every operator in a given algebra of operators.

Throughout the paper,  $X$  is a Banach space. The term “subspace” refers to a norm closed linear subspace of  $X$ , while the term “linear subspace” refers to a subspace that is not necessarily closed. Whenever we say that  $\mathfrak{A}$  is an algebra of operators, we mean that  $\mathfrak{A}$  is an algebra of operators on  $X$ . Also, given a sequence  $(x_i)$ , we write  $[x_i]$  for the closed linear span of  $(x_i)$ .

**Definition 1.2.** Let  $\mathcal{C} \subseteq L(X)$  be an arbitrary collection of operators and  $Y \subseteq X$  a subspace of  $X$ . We call  $Y$  ***almost invariant under***  $\mathcal{C}$ , or  ***$\mathcal{C}$ -almost invariant*** if  $Y$  is almost invariant under every operator in  $\mathcal{C}$ .

Like in the case of a single operator, every subspace that is not a half-space is automatically almost invariant under every collection  $\mathcal{C}$  of operators on  $X$ .

In Section 2, we study the finite-dimensional “errors”  $F$  appearing in formula (1) corresponding to operators in an algebra  $\mathfrak{A}$ . We prove that if  $\mathfrak{A}$  is an algebra without invariant half-spaces then for an  $\mathfrak{A}$ -almost invariant half-space  $Y$  these finite-dimensional subspaces cannot be the same (Proposition 2.2). On the other hand, we prove (Theorem 2.7) that if  $\mathfrak{A}$  is norm closed then these finite-dimensional subspaces cannot be “too far apart”; the dimensions of these subspaces must be uniformly bounded.

In Section 3, the invariant subspaces of algebras having almost invariant half-spaces are investigated. It is proved that if  $\mathfrak{A}$  is a norm closed algebra generated by a single operator then existence of an  $\mathfrak{A}$ -almost invariant half-space implies existence of an  $\mathfrak{A}$ -invariant half-space (Theorem 3.6). This theorem then is generalized to the case of a commutative algebra generated by a finite number of operators (Theorem 3.7). Also, the question of whether the almost invariant half-spaces of  $\mathfrak{A}$  and

$\overline{\mathfrak{A}}^{WOT}$  are the same is investigated (Corollary 3.2 and an example after Theorem 3.6).

In this paper we will occasionally refer to some standard facts about invariant subspaces of operators and algebras of operators. For a general account on invariant subspaces and transitive algebras, see [RR03]. A good review of this topic can be found in [AA02].

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## 2. FINITE-DIMENSIONAL “ERRORS” OF ALMOST INVARIANT HALF-SPACES

Observe that the finite-dimensional subspace  $F$  appearing in the equation (1) is by no means unique. However the minimal dimension of a subspace satisfying this condition is unique. Some simple properties of a subspace of minimal dimension satisfying (1) are collected in the following lemma.

**Lemma 2.1.** *Let  $Y \subseteq X$  be a subspace,  $\mathcal{C}$  be a collection of bounded operators on  $X$  and  $G \subseteq X$  be a finite-dimensional space of the smallest dimension such that  $TY \subseteq Y + G$  for all  $T \in \mathcal{C}$ . Then*

- (i)  $Y + G = Y \oplus G$ ;
- (ii) if  $P : Y \oplus G \rightarrow G$  is the projection along  $Y$  then

$$\text{span} \bigcup_{T \in \mathcal{C}} PT(Y) = G;$$

- (iii) if  $\mathcal{C}$  consists of a single operator, that is  $\mathcal{C} = \{T\}$ , and if  $P : Y \oplus G \rightarrow G$  is the projection along  $Y$  then

$$PT(Y) = G.$$

Moreover, in this case  $G$  can be chosen so that  $G \subseteq TY$ .

*Proof.* (i) Suppose that there exists a non-zero  $g \in G \cap Y$ . Build  $g_2, \dots, g_n \in G$  such that  $\{g, g_2, \dots, g_n\}$  is a basis of  $G$ . Denote  $G_1 =$

$\text{span}\{g_2, \dots, g_n\}$ . It is clear that  $TY \subseteq Y + G_1$  for all  $T \in \mathcal{C}$ . However  $\dim G_1 < \dim G$ .

(ii) Define  $F = \text{span}\{g \in G : v + g \in TY \text{ for some } v \in Y, T \in \mathcal{C}\}$ . Clearly  $F = \text{span} \bigcup_{T \in \mathcal{C}} PT(Y)$ .

We claim that  $TY \subseteq Y + F$  for all  $T \in \mathcal{C}$ . Indeed, if  $y \in Y$  and  $T \in \mathcal{C}$  then  $Ty = v + g$  for some  $v \in Y$  and  $g \in G$ . By definition of  $F$  we get:  $g \in F$ , hence  $Ty \in Y + F$ .

Since  $F \subseteq G$  and  $G$  has the smallest dimension among the spaces with the property  $TY \subseteq Y + G$  for all  $T \in \mathcal{C}$ , we get  $G = F$ .

(iii) The first part of this statement follows immediately from (ii). Let's prove the “moreover” part. Let  $g_1, \dots, g_n$  be a basis of  $G$ . By (ii), there exist  $u_1, \dots, u_n$  and  $y_1, \dots, y_n$  in  $Y$  such that  $Tu_i = y_i + g_i$  ( $i = 1, \dots, n$ ). Put  $f_i = Tu_i$  and  $F = [f_i]_{i=1}^n$ . Then clearly  $F \subseteq TY$ . Also  $Y + F = Y + G$ , so that  $TY \subseteq Y + F$ . From the minimality of  $G$  we obtain that  $\dim F = \dim G$ .  $\square$

The following example shows that  $\bigcup_{T \in \mathcal{C}} PT(Y)$  may not be a linear space even in the case when  $\mathcal{C}$  is an algebra of operators.

*Example.* Let  $X = \ell_2(\mathbb{Z})$ . Define  $T, S \in L(X)$  by

$$Te_0 = e_1, \quad Te_{-1} = e_2, \quad Te_i = 0 \text{ if } i \neq 0, -1,$$

and

$$Se_0 = e_3, \quad Se_i = 0 \text{ if } i \neq 0.$$

Since  $T^2 = S^2 = TS = ST = 0$ , the algebra  $\mathfrak{A}$  generated by  $T$  and  $S$  consists exactly of the operators of form  $aT + bS$  where  $a$  and  $b$  are arbitrary scalars.

Let  $Y = [e_i]_{i \leq 0}$ . Then clearly  $\mathfrak{A}Y \subseteq Y + F$  where  $F = \text{span}\{e_1, e_2, e_3\}$ , and  $F$  is the space of the smallest dimension satisfying this condition. If  $P : Y \oplus F \rightarrow F$  is the projection along  $Y$  then  $\bigcup_{R \in \mathfrak{A}} PR(Y)$  is not a linear space. If it were, it would have been equal to  $F$ , since it contains the basis of  $F$ . However the vector  $e_2 + e_3$  is not in this union.

Suppose  $Y$  is a half-space that is almost invariant under a collection  $\mathcal{C}$  of operators on  $X$ , that is, formula (1) holds for every operator  $T$  in  $\mathcal{C}$  with some  $F$ . One may ask if it is possible that  $F$  does not depend on  $T$ . The following simple reasoning shows that in case of algebras of operators, this can only happen if the algebra already has a common invariant half-space.

**Proposition 2.2.** *Let  $Y \subseteq X$  be a half-space and  $\mathfrak{A}$  an algebra of operators. Suppose that there exists a finite-dimensional space  $F$  such that for each  $T \in \mathfrak{A}$  we have  $TY \subseteq Y + F$ . Then there exists a half-space that is invariant under  $\mathfrak{A}$ .*

*Proof.* Let  $G$  be a space of the smallest dimension such that  $TY \subseteq Y + G$  for all  $T \in \mathfrak{A}$ . We claim that  $Y + G$  is invariant under every operator in  $\mathfrak{A}$ .

Denote  $\mathfrak{A}(Y) = \bigcup_{T \in \mathfrak{A}} TY$ . Clearly  $\mathfrak{A}(Y)$  is invariant under  $\mathfrak{A}$ . Hence, so is  $\text{span } \mathfrak{A}(Y)$ . Denote  $Z = Y + \text{span } \mathfrak{A}(Y)$ . Since  $TY \subseteq \text{span } \mathfrak{A}(Y)$  for every  $T \in \mathfrak{A}$ , we obtain that  $Z$  is invariant under  $\mathfrak{A}$ .

By Lemma 2.1(ii), if  $P : Y \oplus G \rightarrow G$  is a projection along  $Y$  then  $P(\text{span } \mathfrak{A}(Y)) = G$ . Hence  $Y \oplus G = Y \oplus P(\text{span } \mathfrak{A}(Y)) = Y + \text{span } \mathfrak{A}(Y) = Z$ , so that  $Y \oplus G$  is invariant under  $\mathfrak{A}$ .  $\square$

**Definition 2.3.** Let  $T \in L(X)$  be an arbitrary operator and  $Y \subseteq X$  be a linear subspace. We will write  $d_{Y,T}$  for the smallest  $n$  such that there exists  $F$  with  $TY \subseteq Y + F$  and  $\dim F = n$ .

The following observation is obvious.

**Lemma 2.4.** *Let  $T \in L(X)$  be an operator and  $Y \subseteq X$  be a subspace. Let  $q : X \rightarrow X/Y$  be a quotient map. Then  $Y$  is  $T$ -almost invariant if and only if  $(qT)|_Y$  is of finite rank. Moreover,  $\dim(qT)(Y) = d_{Y,T}$ .*

To proceed, we need the following two auxiliary lemmas.

**Lemma 2.5.** *Let  $Y \subseteq X$  be a linear subspace and  $\{u_i\}_{i=1}^N$  be a collection of linearly independent vectors in  $X$  such that  $[u_i]_{i=1}^N \cap Y = \{0\}$ .*

Let  $\{v_i\}_{i=1}^N \subseteq X$  be arbitrary. Then for all but finitely many  $\alpha$  we have  $\{v_i + \alpha u_i\}_{i=1}^N$  is linearly independent and  $[v_i + \alpha u_i]_{i=1}^N \cap Y = \{0\}$ .

*Proof.* Let  $F = \text{span}\{u_i, v_i : i = 1, \dots, N\}$ . Let  $G = (Y + F)/Y$ . Denote  $x_i = u_i + Y \in G$ ,  $z_i = v_i + Y \in G$ . Then the set  $\{x_i\}_{i=1}^N$  is linearly independent. Clearly, to establish the lemma it is enough to prove that the set  $\{z_i + \alpha x_i\}_{i=1}^N$  is linearly independent for all but finitely many  $\alpha$ .

Denote  $M = \dim G$ . Let  $\{b_i\}_{i=1}^M$  be a basis of  $G$  such that  $b_i = x_i$  for all  $1 \leq i \leq N$ . Denote the coordinates of vectors  $z_i$  in this basis by  $z_{ij}$ . Let  $A$  be the  $M \times M$ -matrix with first  $N$  rows consisting of the coordinates of  $z_i$  ( $i = 1, \dots, N$ ), the last  $M - N$  rows being zero rows:

$$A = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1,M-1} & z_{1M} \\ \vdots & \ddots & & & \vdots \\ z_{N1} & z_{N2} & \cdots & z_{N,M-1} & z_{NM} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Since the spectrum of  $A$  is finite,  $\det(A + \alpha I) \neq 0$  for all but finitely many  $\alpha$ . For these  $\alpha$ , the rows of  $A + \alpha I$  must be linearly independent. In particular, the first  $N$  rows are linearly independent. However the first  $N$  rows are exactly the representations of the vectors  $z_i + \alpha x_i$  in the basis  $\{b_i\}_{i=1}^M$ .  $\square$

**Lemma 2.6.** *Let  $Y \subseteq X$  be a linear subspace and  $T \in B(X)$ . Let  $f_1, \dots, f_n \in TY$  be such that no non-trivial linear combination of  $\{f_1, \dots, f_n\}$  belongs to  $Y$ . Then  $n \leq d_{Y,T}$ .*

*Proof.* Let  $q : X \rightarrow X/Y$  be the quotient map. Then  $qf_1, \dots, qf_n$  are linearly independent. Since  $qf_1, \dots, qf_n \in (qT)(Y)$ , we get  $n \leq \dim(qT)(Y) = d_{Y,T}$  by Lemma 2.4.  $\square$

The following theorem is the main statement in this section. Recall that, according to our convention, the term “subspace” stands for a norm closed subspace.

**Theorem 2.7.** *Let  $\mathfrak{S}$  be a subspace of  $L(X)$ . Suppose that  $Y$  is a linear subspace of  $X$  that is almost invariant under  $\mathfrak{S}$ . Then*

$$\sup_{S \in \mathfrak{S}} d_{Y,S} < \infty.$$

*Proof.* For every  $S \in \mathfrak{S}$ , fix a subspace  $F_S \subseteq X$  such that  $SY \subseteq Y + F_S$  and  $\dim F_S = d_{Y,S}$ . By Lemma 2.1,  $Y + F_S$  is a direct sum. Fix  $P_S : Y \oplus F_S \rightarrow F_S$  the projection along  $Y$ . Also fix a basis  $(f_i^S)_{i=1}^{d_{Y,S}}$  of  $F_S$  and a tuple  $(g_i^S)_{i=1}^{d_{Y,S}}$  in  $Y$  such that  $(P_S S)g_i^S = f_i^S$  (this can be done by Lemma 2.1(iii)).

Suppose that the statement of the theorem is not true. Then there exists a sequence of operators  $(S_k) \subseteq \mathfrak{S}$  such that the sequence  $(d_{Y,S_k})_{k=1}^\infty$  is strictly increasing. Without loss of generality,  $\|S_k\| = 1$ .

We will inductively construct a sequence  $(a_k)$  of scalars such that the following two conditions are satisfied for every  $m$ .

- (i) If  $T_m = \sum_{k=1}^m a_k S_k$  then  $N_m := d_{Y,T_m} \geq d_{Y,S_m}$ .
- (ii) Let

$$C_m = \sup_{b_1, \dots, b_{N_m} \in [-1, 1]} \left\| \sum_{i=1}^{N_m} b_i g_i^{T_m} \right\| \cdot \max_{i=1, \dots, N_m} \left\| (f_i^{T_m})^* \right\|,$$

where  $(f_i^{T_m})^*$  is the  $i$ -th biorthogonal functional for  $(f_i^{T_m})_{i=1}^{N_m}$  in  $F_{T_m}^*$ , and

$$D_m = \min \left\{ 1, \frac{1}{C_1 \cdot \|P_{T_1}\|}, \dots, \frac{1}{C_m \cdot \|P_{T_m}\|} \right\}.$$

Then  $0 < a_1 \leq \frac{1}{2}$  and  $0 < a_{m+1} < \frac{1}{2^{m+1}} D_m$  for all  $m \geq 1$ .

Indeed, on the first step put  $a_1 = \frac{1}{2}$ . Suppose that  $a_1, \dots, a_m$  have been constructed. Define  $D_m$  as in (ii). Denote for convenience  $N = d_{Y,S_{m+1}}$ . Let  $u_i = f_i^{S_{m+1}}$  and  $v_i = T_m g_i^{S_{m+1}}$ ,  $i = 1, \dots, N$ . By Lemma 2.5 we can find  $0 < \alpha < \frac{1}{2^{m+1}} D_m$  such that no non-trivial linear combination of vectors from the set  $\{v_i + \alpha u_i\}_{i=1}^N$  is contained in  $Y$ . Put  $a_{m+1} = \alpha$ . This makes both conditions (i) and (ii) satisfied for  $m+1$ . Indeed, condition (ii) is satisfied immediately. Let's check condition (i). Denote for convenience  $y_i = g_i^{S_{m+1}}$ . Observe: for each  $i = 1, \dots, N$ , we have  $T_{m+1} y_i = T_m y_i + \alpha S_{m+1} y_i = v_i + \alpha u_i + w_i$  where

$w_i$  is some vector in  $Y$ . Since no linear combination of  $\{v_i + \alpha u_i\}_{i=1}^N$  is contained in  $Y$ , the same is true for  $\{T_{m+1}y_i\}_{i=1}^N$ . Condition (i) now follows from Lemma 2.6.

Denote  $S = \sum_{k=1}^{\infty} a_k S_k$ . By condition (ii),  $a_k \leq \frac{1}{2^k}$  for all  $k \in \mathbb{N}$ , so that  $S$  is well-defined. For every  $m \in \mathbb{N}$ , denote  $R_m = \sum_{k=m+1}^{\infty} a_k S_k$ , so that  $S = T_m + R_m$ . By condition (ii), we get:  $\|R_m\| < \frac{1}{C_m \cdot \|P_m\|}$  for all  $m \in \mathbb{N}$ .

Clearly,  $S \in \mathfrak{S}$ . By assumptions of the theorem,  $SY = Y \oplus F_S$ . Denote  $n = \dim F_S < \infty$ . Pick  $m \in \mathbb{N}$  such that  $N_m > n$  and put  $z_i = Sg_i^{T_m}$ ,  $i = 1, \dots, N_m$ . Since  $N_m > n$ , there exists a sequence  $(b_i)_{i=1}^{N_m}$  of scalars such that  $\max_i |b_i| = 1$  and

$$z := \sum_{i=1}^{N_m} b_i z_i \in Y.$$

Consider  $y = \sum_{i=1}^{N_m} b_i g_i^{T_m}$ . We have

$$T_m y = S y - R_m y = z - R_m y,$$

hence

$$(P_{T_m} T_m) y = -(P_{T_m} R_m) y.$$

Clearly, for each  $i = 1, \dots, N_m$ , we have  $b_i = (f_i^{T_m})^*(P_{T_m} T_m y)$ . Let  $k$  be such that  $|b_k| = 1$ . Then

$$\begin{aligned} 1 = |b_k| &= \left| (f_k^{T_m})^*(P_{T_m} T_m y) \right| \leq \left\| (f_k^{T_m})^* \right\| \cdot \|P_{T_m} T_m y\| = \\ &= \left\| (f_k^{T_m})^* \right\| \cdot \|P_{T_m} R_m y\| \leq \left\| (f_k^{T_m})^* \right\| \cdot \|P_{T_m}\| \cdot \|R_m\| \cdot \left\| \sum_{i=1}^{N_m} b_i g_i^{T_m} \right\| \leq \\ &\leq \|P_{T_m}\| \cdot \|R_m\| \cdot C_m < \|P_{T_m}\| \frac{1}{C_m \cdot \|P_{T_m}\|} C_m = 1 \end{aligned}$$

which is a contradiction.  $\square$

### 3. INVARIANT SUBSPACES OF ALGEBRAS HAVING ALMOST INVARIANT HALF-SPACES

In this section we study some connections between the invariant subspaces of an algebra of operators and the almost invariant half-spaces

of this algebra. In particular, we establish that if a norm-closed algebra generated by a single operator has an almost invariant half-space then it has an invariant half-space. Then we generalize this to commutative algebras generated by a finite number of operators. Also, we study the question when the almost invariant half-spaces of an algebra and of its WOT-closure are the same.

It is well-known that the invariant subspaces of an algebra of operators coincide with those of the WOT-closure of this algebra. Remarkably, the same statement holds for almost invariant half-spaces, provided that the algebra is norm closed.

**Proposition 3.1.** *Let  $Y$  be a subspace of  $X$  and  $\mathfrak{A}$  an algebra of operators acting on  $X$ . Let  $N \in \mathbb{N}$  be such that  $d_{Y,T} \leq N$  for all  $T \in \mathfrak{A}$ . Then  $d_{Y,T} \leq N$  for all  $T \in \overline{\mathfrak{A}}^{WOT}$ .*

*Proof.* It is enough to prove that if  $T \in \overline{\mathfrak{A}}^{SOT}$  then  $d_{Y,T} \leq N$ . Suppose this is not true. Let  $T \in \overline{\mathfrak{A}}^{SOT}$  be an operator with  $d_{Y,T} \geq N+1$ . Let  $F \subseteq X$  be such that  $\dim F = d_{Y,T}$  and  $TY \subseteq Y \oplus F$ . Fix  $N+1$  linearly independent vectors  $(f_i)_{i=1}^{N+1}$  in  $F$ . By Lemma 2.1(iii), there exist  $(u_i)_{i=1}^{N+1} \subseteq Y$  such that for each  $i = 1, \dots, N+1$  we have  $Tu_i = y_i + f_i$  for some  $y_i \in Y$ . Since  $Y \cap F = \{0\}$ ,  $[Tu_i]_{i=1}^{N+1} \cap Y = \{0\}$  and  $Tu_1, \dots, Tu_{N+1}$  are linearly independent.

Fix a net  $(T_\alpha) \subseteq \mathfrak{A}$  such that  $T_\alpha \xrightarrow{SOT} T$ . Let  $q : X \rightarrow X/Y$  be the quotient map. Since  $[Tu_i]_{i=1}^{N+1} \cap Y = \{0\}$  and  $Tu_1, \dots, Tu_{N+1}$  are linearly independent, the collection  $\{(qT)u_i\}_{i=1}^{N+1}$  is linearly independent. Observe that if  $\varepsilon > 0$  is sufficiently small then each collection  $\{v_i\}_{i=1}^{N+1}$  satisfying  $\|v_i - (qT)u_i\| < \varepsilon$  as  $i = 1, \dots, N+1$  is again linearly independent.

Fix  $\alpha_0$  such that for all  $\alpha \geq \alpha_0$  we have  $\|(T_\alpha - T)(u_i)\| < \varepsilon$  for all  $i = 1, \dots, N+1$ . Then  $\|(qT_\alpha - qT)(u_i)\| < \varepsilon$ , so that the collection  $\{(qT_\alpha)u_i\}_{i=1}^{N+1}$  is linearly independent for all  $\alpha \geq \alpha_0$ . By Lemma 2.4, this, however, implies that  $d_{Y,T_\alpha} \geq N+1$  for all  $\alpha \geq \alpha_0$  which contradicts the assumptions.  $\square$

**Corollary 3.2.** *Let  $\mathfrak{A}$  be a norm closed algebra of operators on  $X$  and  $Y$  be a half-space in  $X$ . Then  $Y$  is  $\mathfrak{A}$ -almost invariant if and only if  $Y$  is  $\overline{\mathfrak{A}}^{WOT}$ -almost invariant.*

*Proof.* If  $Y$  is  $\mathfrak{A}$ -almost invariant then by Theorem 2.7 there exists  $N \in \mathbb{N}$  such that  $d_{Y,S} < N$  for all  $S \in \mathfrak{A}$ . By Proposition 3.1 the same is true for all  $S \in \overline{\mathfrak{A}}^{WOT}$ . This implies that  $Y$  is  $\overline{\mathfrak{A}}^{WOT}$ -almost invariant. The converse statement is evident.  $\square$

We will show later in this section that the condition of  $\mathfrak{A}$  being norm closed is essential here.

The following lemma is standard:

**Lemma 3.3.** *Let  $X$  and  $Y$  be Banach spaces and  $T \in L(X, Y)$  be of finite rank. Then  $\dim(\text{Range } T) = \text{codim}(\ker T)$ .*

We will now introduce some notations. If  $Y$  and  $Z$  are two subspaces of  $X$  and  $Y \subseteq Z$  then the symbol  $\text{codim}_Z Y$  will stand for the codimension of  $Y$  in  $Z$ .

Let  $T \in L(X)$  be an operator and  $Y \subseteq X$  be a half-space. Consider two procedures of constructing new linear spaces:

$$\begin{aligned} D_T(Y) &= \{y \in Y : Ty \in Y\} && \text{"going downwards"}, \\ U_T(Y) &= Y + TY && \text{"going upwards"}. \end{aligned}$$

Clearly  $D_T(Y) \subseteq Y \subseteq U_T(Y)$ .

**Lemma 3.4.** *Let  $Y \subseteq X$  be a half-space and  $T \in L(X)$ . If  $Y$  is  $T$ -almost invariant then both  $D_T(Y)$  and  $U_T(Y)$  are half-spaces. Moreover,  $\text{codim}_Y D_T(Y) = \text{codim}_{U_T(Y)} Y = d_{Y,T}$ .*

*Proof.* The statement about  $U_T(Y)$  follows immediately from the definition of an almost invariant subspace. Let's verify the statement about  $D_T(Y)$ . Clearly we only need to verify the "moreover" part.

Let  $TY \subseteq Y+F$  where  $F$  is such that  $\dim F = d_{Y,T}$ . By Lemma 2.1(i), we have  $Y+F = Y \oplus F$ . Let  $P : Y \oplus F \rightarrow F$  be the projection along  $Y$ . Then  $D_T(Y) = \ker(PT|_Y)$ . By Lemma 3.3 we get  $\text{codim}_Y D_T(Y) = \dim(\text{Range } PT|_Y) = \dim F = d_{Y,T}$ .  $\square$

The following lemma is the key statement of this chapter.

**Lemma 3.5.** *Suppose that  $Y$  is a half-space in  $X$  that is almost invariant under an operator  $T \in L(X)$ . If  $d_{Y,T} > 0$  and  $d_{D_T^k(Y),T} \geq d_{Y,T}$  and  $d_{U_T^k(Y),T} \geq d_{Y,T}$  for all  $k \in \mathbb{N}$  then  $d_{Y,T^m} \geq m$  for all  $m \in \mathbb{N}$ .*

*Proof.* Denote for convenience  $N = d_{Y,T}$ . Fix  $f_1^0, \dots, f_N^0 \in X$  such that  $TY \subseteq Y \oplus F$  where  $F = [f_i^0]_{i=1}^N$ . In particular,  $\dim F = d_{Y,T}$ .

Suppose that  $d_{D_T^k(Y),T} \geq N$  and  $d_{U_T^k(Y),T} \geq N$  for all  $k \in \mathbb{N}$ . Denote  $Y_0 = Y$  and  $Y_k = D_T^k(Y)$ ,  $k \geq 1$ . Since  $Y_k = D_T(Y_{k-1})$  for all  $k \geq 1$ , it follows that  $TY_k \subseteq Y_{k-1}$  as  $k \geq 1$ .

We claim that for each  $k \geq 1$  there exists an  $N$ -tuple  $(f_1^k, \dots, f_N^k)$  in  $Y_{k-1}$  such that

- (i)  $Y_k \oplus F_k = Y_{k-1}$  where  $F_k = [f_i^k]_{i=1}^N$ , and
- (ii) if  $P_k : Y_k \oplus F_k \rightarrow F_k$  is the projection along  $Y_k$  then  $(P_{k-1}T)f_i^k = f_i^{k-1}$  for all  $i = 1, \dots, N$  (if  $k = 1$  then we assume  $P_0 : Y \oplus F \rightarrow F$  is the projection along  $Y$ ).

Let  $k = 1$ . By Lemma 2.1(iii), for each  $i = 1, \dots, N$ , we can find  $f_i^1 \in Y$  such that  $(P_0T)f_i^1 = f_i^0$ . Then (ii) is satisfied. Write  $F_1 = [f_i^1]_{i=1}^N$ . Since  $Y_1 \cap F_1 = \{0\}$  by definition of  $Y_1$  and  $\dim F_1 = N = \text{codim}_Y Y_1$  by Lemma 3.4,  $Y_1 \oplus F_1 = Y$ .

Suppose the claim is true for  $k \geq 0$ . Then  $Y_k \oplus F_k = Y_{k-1}$ . Since  $TY_k \subseteq Y_{k-1}$  and  $d_{Y_k,T} \geq N = \dim F_k$ , we get  $d_{Y_k,T} = N$ . Then from Lemma 2.1(iii) for each  $i = 1, \dots, N$  there exists  $f_i^{k+1} \in Y_k$  such that  $(P_kT)f_i^{k+1} = f_i^k$ , so that (ii) is satisfied for  $k+1$ . To show (i), write  $F_{k+1} = [f_i^{k+1}]_{i=1}^N$  and observe:  $Y_{k+1} \cap F_{k+1} = \{0\}$  by definition of  $Y_{k+1}$  and  $\dim F_{k+1} = N = \text{codim}_{Y_k} Y_{k+1}$  by Lemma 3.4.

Observe that from condition (ii) of this claim we have: for each  $k \geq 1$  there exists  $y \in Y$  such that  $T^k f_i^k = y + f_i^0$ . That is,  $f_i^k$  is a  $k$ -th “preimage” of  $f_i^0$ . It follows that any  $f \in F$  has a  $k$ -th “preimage” in  $Y_{k-1}$ .

Denote  $Z_0 = Y$ ,  $Z_k = U_T^k(Y)$ ,  $k \geq 1$ . That is,  $Z_k = U_T(Z_{k-1})$ . In particular,  $TZ_{k-1} \subseteq Z_k$  for all  $k \geq 0$ . We claim that  $Z_k = Y \oplus F \oplus$

$TF \cdots \oplus T^{k-1}F$ . Indeed, for  $k = 0$  this is obvious. Suppose the claim is true for  $k \geq 1$ . Let's prove that  $Z_{k+1} = Y \oplus F \oplus TF \cdots \oplus T^kF$ . We have  $Z_{k+1} = U_T(Z_k) = Z_k + TZ_k = (Y \oplus F \oplus TF \cdots \oplus T^{k-1}F) + (TY + TF + T^2F + \cdots + T^kF) = (Y \oplus F \oplus TF \cdots \oplus T^{k-1}F) + T^kF$  since  $TY \subseteq Y \oplus F$ . That is,  $Z_{k+1} = Z_k + T^kF$ . We only have to prove that this sum is direct. We have  $\dim T^kF \leq N$  since  $\dim F = N$ . On the other hand,  $TZ_k \subseteq Z_{k+1} = Z_k + T^kF$ . Since  $d_{Z_k, T} \geq N$  for all  $k \geq 0$ , we get  $\dim T^kF = N = d_{Z_k, T}$ . By Lemma 2.1(i), the sum must be direct.

Observe that in particular, this means that if  $f \in F$  is non-zero then  $T^k f \in Z_{k+1} \setminus Z_k$  ( $k \geq 0$ ).

Let  $u \in F$  be a non-zero vector,  $m \in \mathbb{N}$  be arbitrary, and  $k \in \{1, \dots, m\}$ . Put  $u_k$  to be the  $k$ -th “preimage” of  $u$ , that is, such a vector in  $Y_{k-1}$  that  $T^k u_k = v_k + u$  for some  $v_k \in Y$ . Then  $T^m u_k = T^{m-k} T^k u_k = T^{m-k} v_k + T^{m-k} u$ . Since  $v_k \in Y$ , it follows that  $T^{m-k} v_k \in Z_{m-k}$ . Also since  $u \neq 0$ , we get  $T^{m-k} u \in Z_{m-k+1} \setminus Z_{m-k}$ . Since  $Z_{m-k} \subseteq Z_{m-k+1}$  we have  $T^m u_k \in Z_{m-k+1} \setminus Z_{m-k}$ .

This means that  $T^m Y$  contains  $m$  vectors  $\{T^m(u_k)\}_{k=1}^m$  such that no non-zero linear combination of these vectors belongs to  $Y$ . By Lemma 2.6 we get:  $d_{Y, T^m} \geq m$ .  $\square$

As an immediate corollary we get:

**Theorem 3.6.** *Let  $T \in L(X)$  be an operator and  $\mathfrak{A}$  the norm closed algebra generated by  $T$ . If  $\mathfrak{A}$  has an almost invariant half-space then  $\mathfrak{A}$  has an invariant half-space.*

*Proof.* If  $d_{Y, T} = 0$  then there is nothing to prove. Let  $d_{Y, T} > 0$ . Since  $\mathfrak{A}$  is norm closed,  $\sup_{S \in \mathfrak{A}} d_{Y, S} < \infty$  by Theorem 2.7. In particular,  $\sup_{m \in \mathbb{N}} d_{Y, T^m} < \infty$ . By Lemma 3.5 we obtain that either  $d_{D_T^k(Y), T} < d_{Y, T}$  or  $d_{U_T^k(Y), T} < d_{Y, T}$  for some  $k \in \mathbb{N}$ .

Applying this finitely many times we get a half-space  $Z$  such that  $d_{Z, T} = 0$ . Since  $Z$  is  $T$ -invariant, it is  $\mathfrak{A}$ -invariant.  $\square$

This theorem allows us to get an earlier promised example of a (not necessarily closed) algebra  $\mathfrak{A}$  whose almost invariant half-spaces are different from those of  $\overline{\mathfrak{A}}^{WOT}$  (and even from those of the norm closure of  $\mathfrak{A}$ ). This example also shows that, unlike in case of invariant subspaces, there exists an operator whose almost invariant half-spaces are different from those of the norm closed algebra generated by this operator.

*Example.* Let  $D$  be a Donoghue operator on  $\ell_2$ . That is,  $D$  is a backward shift with non-zero weights  $(w_n)_{n=1}^\infty$  which satisfy conditions ( $|w_n|_{n=1}^\infty$  is monotone decreasing and is in  $\ell_2$  (see, e.g., [RR03, Section 4.4] for the properties of Donoghue operators). Put  $\mathfrak{A} = \{p(D) : p \text{ is a polynomial such that } p(0) = 0\}$ . It was proved in [APTT] that  $D$  has an almost invariant half-space. Then  $\mathfrak{A}$  has an almost invariant half-space. However all the invariant subspaces of  $D$  are finite dimensional (see [RR03, Theorem 4.12]) and therefore  $D$  has no invariant half-spaces. By Theorem 3.6,  $\overline{\mathfrak{A}}^{\|\cdot\|}$  has no almost invariant half-spaces.

The following result is a generalization of Theorem 3.6.

**Theorem 3.7.** *Let  $\mathfrak{A}$  be a norm-closed algebra generated by a finite number of pairwise commuting operators. If  $\mathfrak{A}$  has an almost invariant half-space then  $\mathfrak{A}$  has an invariant half-space.*

*Proof.* Let  $\mathfrak{A}$  be generated by pairwise commuting operators  $T_1, \dots, T_n$  and let  $Y$  be an  $\mathfrak{A}$ -almost invariant half-space. We will prove that there exists a half-space that is invariant under  $T_k$  for each  $k = 1, \dots, n$ .

First, observe that if  $T \in \mathfrak{A}$  then both  $D_T(Y)$  and  $U_T(Y)$  are  $\mathfrak{A}$ -almost invariant because  $\text{codim}_Y D_T(Y) < \infty$  and  $\text{codim}_{U_T(Y)} Y < \infty$  by Lemma 3.4. Next, we claim that if  $S \in \mathfrak{A}$  is such that  $Y$  is  $S$ -invariant then  $D_T(Y)$  and  $U_T(Y)$  are again  $S$ -invariant. Indeed, let  $y \in D_T(Y)$ , then  $T(Sy) = STy \in Y$  since  $Ty \in Y$  and  $Y$  is  $S$ -invariant. Hence  $Sy \in D_T(Y)$ , so that  $D_T(Y)$  is  $S$ -invariant. Let  $u + v \in U_T(Y)$ , with  $u \in Y$  and  $v \in Ty$  for some  $y \in Y$ . Then

$S(u+v) = Su + STy = Su + TSy \in U_T(Y)$  since  $Su, Sy \in Y$ , so that  $U_T(Y)$  is  $S$ -invariant.

For  $k = 1, \dots, n$ , denote by  $\mathfrak{A}_k$  the norm closed algebra generated by  $T_k$ . Clearly  $\mathfrak{A}_k \subseteq \mathfrak{A}$  and hence every  $\mathfrak{A}$ -almost invariant half-space is  $\mathfrak{A}_k$ -almost invariant for all  $k = 1, \dots, n$ .

Apply a finite sequence of procedures  $D_{T_1}$  and  $U_{T_1}$  to  $Y$  to obtain a  $T_1$ -invariant half-space  $Y_1$ , as in the proof of Theorem 3.6. By the discussion above,  $Y_1$  is  $\mathfrak{A}$ -almost invariant. Apply a finite sequence of procedures  $D_{T_2}$  and  $U_{T_2}$  to  $Y_1$  to obtain a  $T_2$ -invariant half-space. Then  $Y_2$  is  $T_1$ - and  $T_2$ -invariant and still  $\mathfrak{A}$ -almost invariant. Repeat this procedure  $n - 2$  more times to get an  $\mathfrak{A}$ -invariant half-space.  $\square$

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